Precision calculation of the $\pi^- d$ scattering length

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Outline

Introduction

- πN scattering lengths
- Hadronic atoms

Few-body contributions to $a_{\pi^- d}$

- Strong diagrams
- Virtual photons
- Dispersive and Δ corrections

Results

- Combined analysis of πH and πD
- Goldberger–Miyazawa–Oehme sum rule

arXiv:1003.4444

πN scattering lengths

- In the isospin limit

$$T^{ab}_{\pi N}(0) = 4\pi\left(1+\frac{M_\pi}{m_\mathrm{p}}\right)\chi^\dagger_{N'}\left(a^+\delta^{ab}+a^-\frac{1}{2}[\tau^a,\tau^b]\right)\chi_N$$

- Isovector and isoscalar scattering lengths

Weinberg, 1966

$$a^- \sim \frac{M_\pi}{8\pi F_\pi^2} \sim 90\cdot 10^{-3} M_\pi^{-1}, \qquad a^+ \sim 0$$

πN scattering lengths

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- Isovector and isoscalar scattering lengths Weinberg, 1966

$$a^- \sim \frac{M_\pi}{8\pi F_\pi^2} \sim 90\cdot 10^{-3} M_\pi^{-1}, \qquad a^+ \sim 0$$

- Precise determination of a^+ difficult
- Aim: extract a^+ and a^- from hadronic atoms with high accuracy
- πNN coupling constant g_c via the GMO sum rule
- Dispersive analysis of the πN σ-term

Hadronic atoms: πH and πD

- πH and πD bound by electromagnetism ($e^- \to \pi^-$)
- Strong interaction modifies the spectrum
- Shift of the ground state in πH Lyubovitskij, Rusetsky, 2000

$$\epsilon_{1s} = E_{1s} - E_{1s}^{\text{QED}} = -2\alpha^3\mu_H^2 \underbrace{\left(a^+ + a^- + \Delta a_{\pi^- p}\right)}_{a_{\pi^- p}} (1 + K_\epsilon + \delta_\epsilon^{\text{vac}})$$

$$K_\epsilon = 2\alpha(1 - \log\alpha)\mu_H a_{\pi^- p}, \qquad \delta_\epsilon^{\text{vac}} = 2\frac{\delta\Psi_H(0)}{\Psi_H(0)} = 0.48\,\%$$

- Width of the ground state yields information on a^- ($\pi^- p \to \pi^0 n$, $\pi^- p \to \gamma n$)

Hadronic atoms: πH and πD

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- Width of the ground state yields information on a^-
($\pi^- p \to \pi^0 n$, $\pi^- p \to \gamma n$)
- Level shift in πD sensitive to

$$\text{Re}\, a_{\pi^- d} = \frac{2(1 + M_\pi/m_{\text{p}})}{1 + M_\pi/m_{\text{d}}}(a^+ + \Delta a^+) + a_{\pi^- d}^{(3)} \sim -25 \cdot 10^{-3} M_\pi^{-1}$$

- Three constraints for $a^\pm \Rightarrow$ systematics, accuracy
- Isospin breaking and $a_{\pi^- d}^{(3)}$ need to be well under control!

Isospin violation and few-body effects

- a^+ itself cannot be determined from πH and πD, only

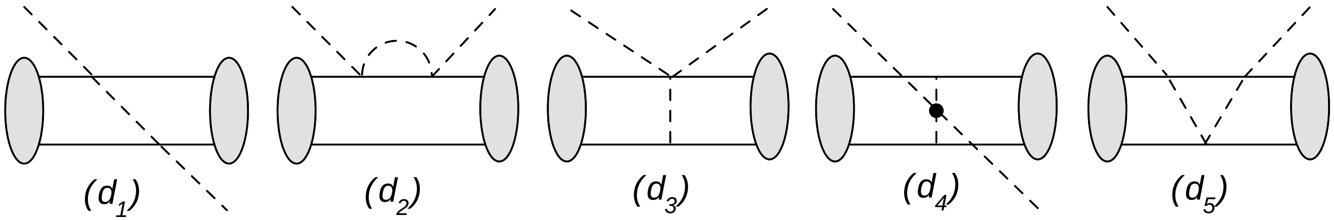
$$\tilde{a}^+ = a^+ + \frac{1}{4\pi(1+M_\pi/m_\mathrm{p})}\left\{\frac{4(M_\pi^2 - M_{\pi^0}^2)}{F_\pi^2}c_1 - 2e^2 f_1\right\}$$

- Large 2B IV corrections (NLO in ChPT) MH, Kubis, Meißner, 2009

$$a_{\pi^-p} + a_{\pi^-n} = 2(\tilde{a}^+ + \Delta\tilde{a}^+), \qquad \Delta\tilde{a}^+ = (-3.3 \pm 0.3)\cdot 10^{-3} M_\pi^{-1}$$

- Requires $a^{(3)}_{\pi^-d}$ with an accuracy of better than $10\,\%$ $\Rightarrow$ need to worry about isospin violation in 3B sector

Strong diagrams



- Strong contributions in ChPT extensively discussed in the literature Weinberg, 1992, Beane *et al.*, 2003, Baru *et al.*, 2004, Liebig *et al.*, 2010
- Accuracy limited by unknown contact term at NNLO, estimate

$$p^{3/2} \sim (M_\pi/m_\mathrm{p})^{3/2} \sim 6\,\%$$

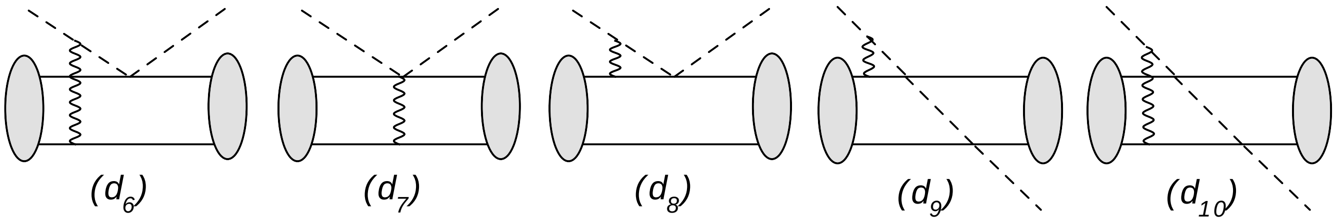
- New: isospin-violating corrections to (d_1)–(d_4) due to mass differences and in the πN scattering lengths
- Result:

$$a^\mathrm{str} = (-22.6 \pm 1.1 \pm 0.4) \cdot 10^{-3} M_\pi^{-1}$$

CD-Bonn, AV18, NNLO χEFT

- Wave-function dependence $\sim 5\,\%$

Virtual photons



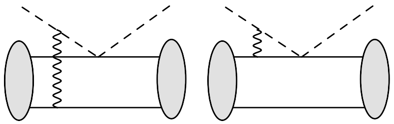
- New scales: $\sqrt{M_\pi \epsilon}$ (three-body dynamics) and $\sqrt{m_\mathrm{p}\epsilon}$ (deuteron wave function)
- (d_6) and (d_8)–(d_{10}) "would-be infrared singular"

$$\left\langle \frac{1}{\mathbf{q}^2(\mathbf{q}^2+\delta)} \right\rangle, \quad \delta = 2\sqrt{M_\pi^2+\mathbf{q}^2}\left(\epsilon + \frac{\mathbf{p}^2+(\mathbf{p}-\mathbf{q})^2}{2m_\mathrm{p}}\right)$$

$\Rightarrow$ potentially enhanced by $\sqrt{M_\pi/\epsilon}$

- χPT counting applicable in infrared enhanced diagrams?
- Separate those contributions already included in the Deser formula (deuteron pole)

Virtual photons: isovector case



$$\sim \int d^3p\, d^3q \left(\Psi^\dagger(\mathbf{p}) - \Psi^\dagger(\mathbf{p}-\mathbf{q})\right)\Psi(\mathbf{p})\frac{1}{\mathbf{q}^2(\mathbf{q}^2+\delta)} \equiv I^{(d_8)} - I^{(d_6)}$$

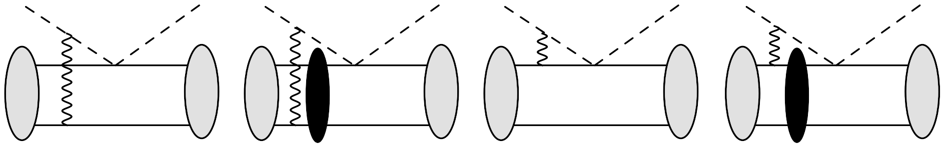
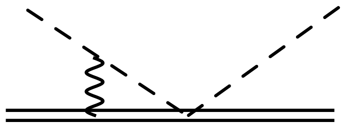
- Pauli principle: $(-1)^{L+T+S} = -1 \Rightarrow$ no S-wave NN interaction
- At leading order:

$$I^{(d_6)} = I^{(d_8)} = \int d^3p\, d^3q \frac{\Psi^\dagger(\mathbf{p})\Psi(\mathbf{p})}{\mathbf{q}^2\left(\mathbf{q}^2 + 2M_\pi(\epsilon + \mathbf{p}^2/m_\mathrm{p})\right)}$$

$$= 2\pi^2\sqrt{\frac{m_\mathrm{p}}{2M_\pi}}\int d^3p \frac{\Psi^\dagger(\mathbf{p})\Psi(\mathbf{p})}{\sqrt{\mathbf{p}^2 + m_\mathrm{p}\epsilon}} \sim \frac{8\pi}{3\sqrt{2}}\frac{1}{\sqrt{M_\pi \epsilon}}, \qquad \Psi(\mathbf{p}) \sim \frac{\sqrt[4]{m_\mathrm{p}\epsilon}/\pi}{\mathbf{p}^2 + m_\mathrm{p}\epsilon}$$

$\Rightarrow$ potentially enhanced contributions cancel exactly

Virtual photons: isoscalar case

$a_{T=0}^{(d_6)+(d_8)} =$  $-$ 

$$= \frac{4e^2(1+M_\pi/m_{\rm p})}{1+M_\pi/m_{\rm d}} a^+ \int \frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2}\Bigg\{\frac{|F(\mathbf{k})|^2-1}{\mathbf{k}^2/2M_\pi - i\eta}$$

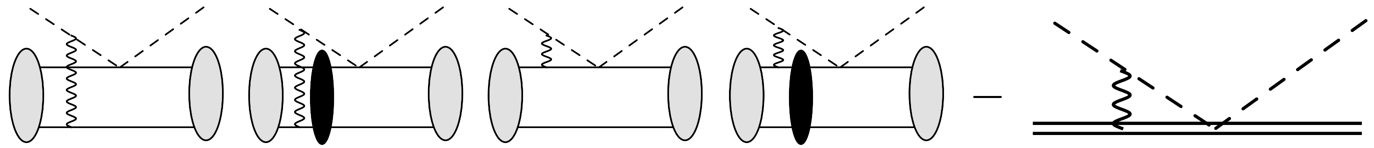
$$+\int \frac{d^3p}{(2\pi)^6}\frac{1}{\epsilon+\mathbf{p}^2/m_{\rm p}+\mathbf{k}^2/2M_\pi - i\eta} G^s_{\mathbf{p}}(\mathbf{k})\frac{1}{2}\left(G^s_{\mathbf{p}}(\mathbf{k})+G^s_{\mathbf{p}}(-\mathbf{k})\right)\Bigg\}$$

$$F(\mathbf{k}) = \int d^3q\Psi^\dagger(\mathbf{q})\Psi(\mathbf{q}-\mathbf{k}/2) \quad \Rightarrow \quad |F(\mathbf{k})|^2-1=\mathcal{O}(\mathbf{k}^2)$$

$$G^s_{\mathbf{p}}(\mathbf{k}) = \int d^3q\Psi^\dagger(\mathbf{q})\Psi^s_{\mathbf{p}}(\mathbf{q}-\mathbf{k}/2) = \mathcal{O}(\mathbf{k})$$

- S-wave NN interaction allowed
- Need to subtract the (infrared divergent) deuteron pole

Virtual photons: isoscalar case

$a_{T=0}^{(d_6)+(d_8)} =$ 

$$= \frac{4e^2(1+M_\pi/m_\mathrm{p})}{1+M_\pi/m_\mathrm{d}} a^+ \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2} \Bigg\{ \frac{|F(\mathbf{k})|^2-1}{\mathbf{k}^2/2M_\pi - i\eta}$$

$$+ \int \frac{d^3p}{(2\pi)^6} \frac{1}{\epsilon + \mathbf{p}^2/m_\mathrm{p} + \mathbf{k}^2/2M_\pi - i\eta} G^s_\mathbf{p}(\mathbf{k}) \frac{1}{2}\big(G^s_\mathbf{p}(\mathbf{k}) + G^s_\mathbf{p}(-\mathbf{k})\big) \Bigg\}$$

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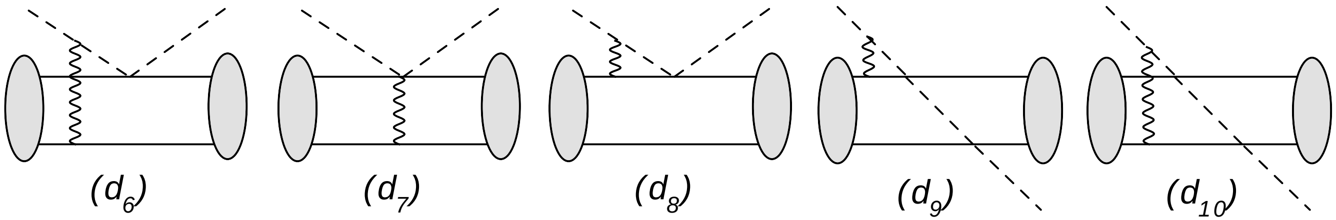
$$G^s_\mathbf{p}(\mathbf{k}) = \int d^3q \Psi^\dagger(\mathbf{q})\Psi^s_\mathbf{p}(\mathbf{q}-\mathbf{k}/2) = \mathcal{O}(\mathbf{k})$$

- S-wave NN interaction allowed
- Need to subtract the (infrared divergent) deuteron pole
- Use orthogonality of deuteron and continuum wave functions
- No infrared enhanced contributions from $p \sim \sqrt{M_\pi \epsilon}$ altogether!
- Isoscalar part may be dropped due to the suppression of a^+

Virtual photons



- Contributions from $p \sim \sqrt{m_{\rm p}\epsilon}$ calculated explicitly $\Rightarrow$ negligible
- Remaining effect from momenta of order M_π

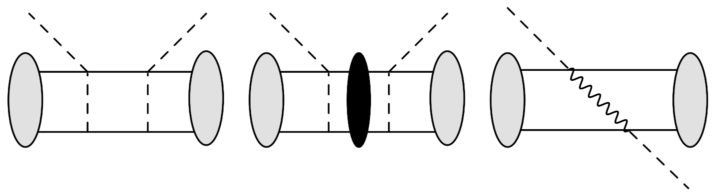
$$a^{\rm EM} = (0.95 \pm 0.01) \cdot 10^{-3} M_\pi^{-1}$$

$\Rightarrow$ residual isovector contributions (and (d_7))

- Consistent with power-counting estimate
- Same conclusions for the higher-order diagrams (d_9) and (d_{10})

Dispersive and Δ corrections

Dispersive corrections: Lensky *et al.*, 2006

$a^{\text{disp}} =$  $+ \cdots = (-2.9 \pm 1.4) \cdot 10^{-3} M_\pi^{-1}$

- Associated with $\pi^- d \to NN \to \pi^- d$ and $\pi^- d \to NN\gamma \to \pi^- d$
- Momenta of order $p \sim \sqrt{M_\pi m_\text{p}} \Rightarrow$ threshold for $NN \to NN\pi$

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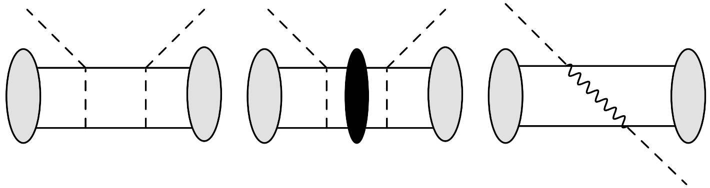
$a^{\text{disp}} =$  $+ \cdots = (-2.9 \pm 1.4) \cdot 10^{-3} M_\pi^{-1}$

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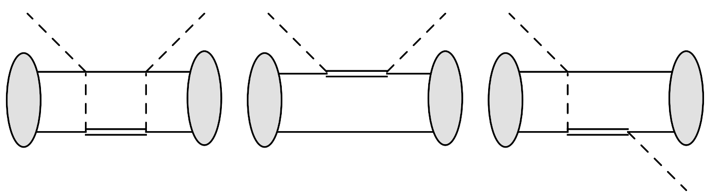
Δ corrections: Baru *et al.*, 2007

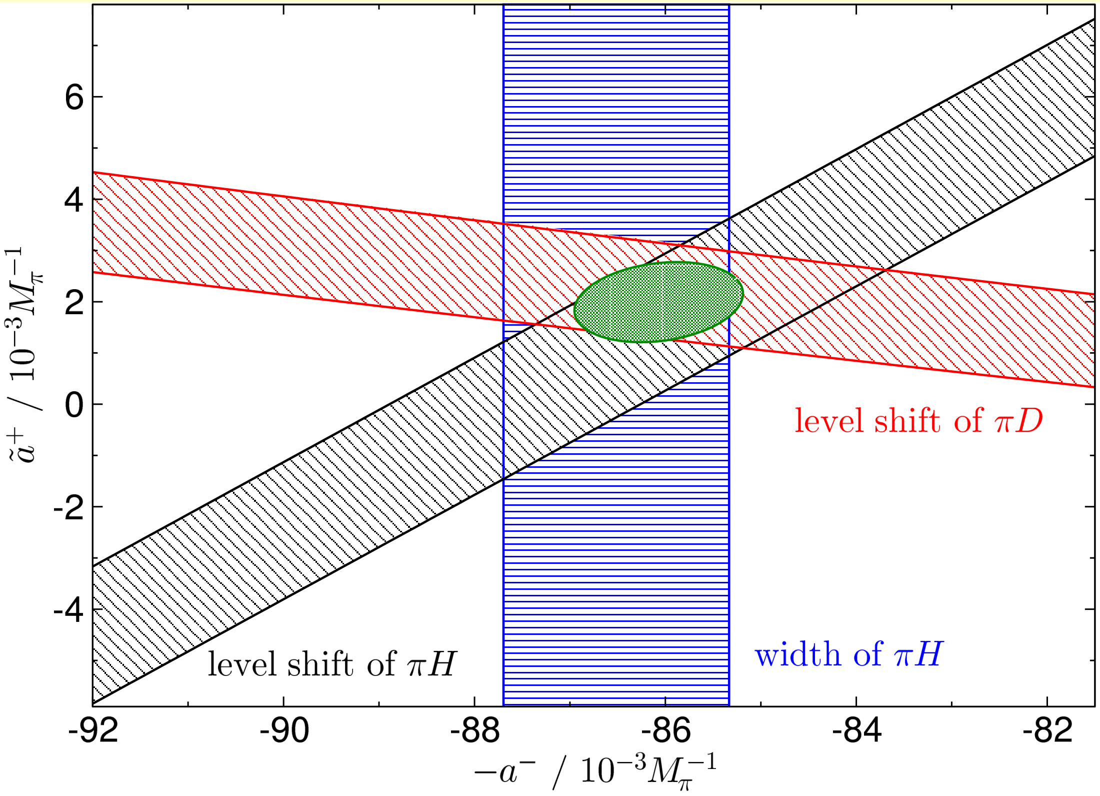
$a^{\Delta} =$  $+ \cdots = (2.4 \pm 0.4) \cdot 10^{-3} M_\pi^{-1}$

- Momenta of order $p \sim \sqrt{(m_\Delta - m_{\text{p}} - M_\pi) m_{\text{p}}} \sim \sqrt{M_\pi m_{\text{p}}}$

Both suppressed by $p^{3/2}$ relative to double scattering (d_1)

$$a^{\text{disp}+\Delta} = (-0.6 \pm 1.5) \cdot 10^{-3} M_\pi^{-1}$$

Combined analysis of πH and πD



Experiment:

Gotta *et al.*, 2005, 2010

πN scattering lengths (with $c_1 = -(1.0 \pm 0.3)\,\mathrm{GeV}^{-1},\ |f_1| \leq 1.4\,\mathrm{GeV}^{-1}$)

$$a^+ = (7.7 \pm 3.1) \cdot 10^{-3} M_\pi^{-1}, \qquad \tilde{a}^+ = (2.0 \pm 0.8) \cdot 10^{-3} M_\pi^{-1}$$

$$a^- = (86.1 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$$

Goldberger–Miyazawa–Oehme sum rule

- Fixed-t dispersion relations at threshold $\Rightarrow$ GMO sum rule

$$\frac{g_c^2}{4\pi} = \left(\left(\frac{m_\mathrm{p}+m_\mathrm{n}}{M_\pi}\right)^2 - 1\right)\left\{\left(1+\frac{M_\pi}{m_\mathrm{p}}\right)\frac{M_\pi}{4}\left(a_{\pi^-p} - a_{\pi^+p}\right) - \frac{M_\pi^2}{2}J^-\right\}$$

$$J^- = \frac{1}{4\pi^2}\int_0^\infty dk \frac{\sigma^{\mathrm{tot}}_{\pi^-p}(k) - \sigma^{\mathrm{tot}}_{\pi^+p}(k)}{\sqrt{M_\pi^2+k^2}}$$

- J^- investigated in detail by Ericson *et al.*, 2002, Abaev *et al.*, 2007

$$J^- = (-1.073 \pm 0.034)\,\mathrm{mb}$$

- With the scattering lengths from πH and πD

$$\frac{g_c^2}{4\pi} = 13.68 \pm 0.12 \pm 0.15 = 13.68 \pm 0.20$$

- In agreement with $g_c^2/4\pi = 13.54 \pm 0.05$ (NN) de Swart *et al.*, 1997 and $g_c^2/4\pi = 13.75 \pm 0.15$ (πN) Arndt *et al.*, 1994

Conclusions

- Calculation of a_{π^-d} at the same accuracy as isospin-violating two-body corrections
- Higher accuracy requires knowledge of the NNLO $(N^\dagger N)^2\pi^\dagger\pi$ contact term
- No infrared-enhanced photon contributions
- Constraints on the πN scattering lengths from a combined analysis of πH and πD
- a^+ positive with 2σ significance
- $g_c^2/4\pi$ from GMO sum rule consistent with NN and πN phase-shift analyses

Spares

Deser formulae

- Width of πH — Zemp, 2004

$$\Gamma_{1s} = 4\alpha^3\mu_H^2 p_1 \left(1+\frac{1}{P}\right)\Big(\underbrace{-\sqrt{2}\,a^- + \Delta a^{\text{cex}}_{\pi^- p}}_{a_{\pi^- p\to\pi^0 n}\equiv a^{\text{cex}}_{\pi^- p}}\Big)^2 (1+K_\Gamma+\delta^{\text{vac}}_\epsilon)$$

$$K_\Gamma = 4\alpha(1-\log\alpha)\mu_H a_{\pi^- p} + 2\mu_H\big(m_\text{p}+M_\pi-m_\text{n}-M_{\pi^0}\big)(a_{\pi^0 n})^2$$

p_1: CMS momentum of $\pi^0 n$, $\quad P = \frac{\sigma(\pi^- p\to\pi^0 n)}{\sigma(\pi^- p\to\gamma n)} = 1.546\pm 0.009$

- Level shift of πD — Meißner *et al.*, 2005

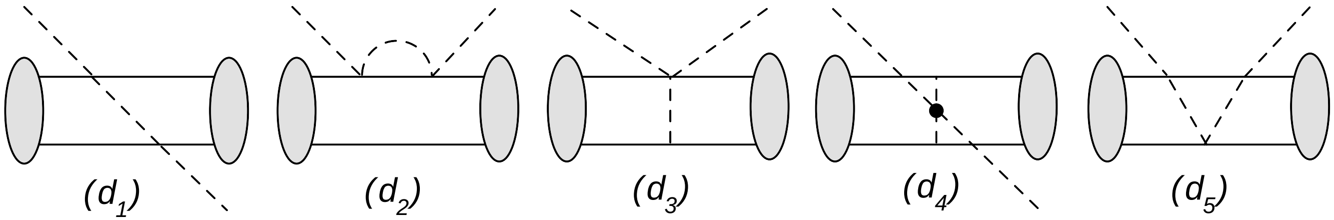
$$\epsilon^D_{1s} = -2\alpha^3\mu_D^2 \,\text{Re}\, a_{\pi^- d}\,(1+K_D+\delta^{\text{vac}}_{\epsilon D})$$

$$\text{Re}\, a_{\pi^- d} = \frac{2(1+M_\pi/m_\text{p})}{1+M_\pi/m_\text{d}}(a^+ + \Delta a^+) + a^{(3)}_{\pi^- d}$$

$$K_D = 2\alpha(1-\log\alpha)\mu_D \,\text{Re}\, a_{\pi^- d}, \qquad \delta^{\text{vac}}_{\epsilon D} = 0.51\,\%$$

- Width of πD: BR $(\pi^- d\to nn)\sim 74\,\%$, $\quad$ BR $(\pi^- d\to nn\gamma)\sim 26\,\%$
- Experiment: $\epsilon_{1s} = (-7.120\pm 0.012)\,\text{eV}$, $\quad \Gamma_{1s} = (0.823\pm 0.019)\,\text{eV}$, $\epsilon^D_{1s} = (2.325\pm 0.031)\,\text{eV}$ — Gotta *et al.*, 2005, 2010

Strong diagrams

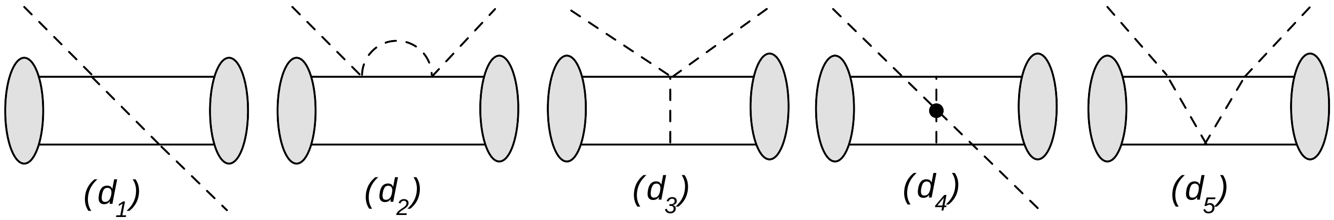


- Power counting relative to (d_1): (modified) Weinberg counting
  Liebig *et al.*, 2010
- Contact term with unknown coefficient at $\mathcal{O}(p^2)$, $p \sim M_\pi/m_\mathrm{p}$
  $\Rightarrow$ include all terms up to $\mathcal{O}(p^{3/2})$, accuracy estimate $\leq 6\,\%$
- Leading contribution: static one-pion-exchange Weinberg, 1992

$$a^{(d_1)} = -\frac{2(1+M_\pi/m_\mathrm{p})^2(a^-)^2}{\pi^2(1+M_\pi/m_\mathrm{d})}\left\langle\frac{1}{\mathbf{q}^2}\right\rangle$$

- NLO vanishes Beane *et al.*, 2003
- πNN cut could lead to enhancement by $\sqrt{m_\mathrm{p}/M_\pi}$
- Any enhanced contribution cancels between (d_1) and (d_2) (embedded in the πNN system) Baru *et al.*, 2004

Strong diagrams



- Isospin-violating corrections to (d_1)–(d_4) due to mass differences and in the πN scattering lengths
- (d_5) formally of $\mathcal{O}(p^2)$, but enhanced by π^2 (Coulombic propagators) $\Rightarrow$ needs to be included
- Similar enhancements possible for all terms of the multiple-scattering series, but

$$\text{full series} - (d_1) - (d_5) \sim 0.1 \cdot 10^{-3} M_\pi^{-1}$$

- Result:

$$a^{\rm str} = (-22.6 \pm 1.1 \pm 0.4) \cdot 10^{-3} M_\pi^{-1}$$

CD-Bonn, AV18, NNLO χEFT

- Wave-function dependence $\sim 5\,\%$

One pion exchange

$$a^{\mathrm{OPE}} = a^{\mathrm{static}} + a^{\mathrm{static}}_{\mathrm{NLO}} + a^{\mathrm{cut}} + \Delta a^{(2)}$$

$$a^{\mathrm{static}} = -\bar{a}^2 \int d^3p\, d^3q\, \Psi^\dagger(\mathbf{p}-\mathbf{q})\Psi(\mathbf{p}) \frac{1}{\mathbf{q}^2}$$

$$a^{\mathrm{static}}_{\mathrm{NLO}} = \bar{a}^2 \int d^3p\, d^3q\, \Psi^\dagger(\mathbf{p}-\mathbf{q})\Psi(\mathbf{p}) \frac{1}{\mathbf{q}^2}\left(\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{q}}+m_{\mathrm{p}}}\right)$$

$$a^{\mathrm{cut}} = \bar{a}^2 \int d^3p\, d^3q \left(\Psi^\dagger(\mathbf{p}-\mathbf{q}) - \Psi^\dagger(\mathbf{p})\right)\Psi(\mathbf{p})\left(\frac{1}{\mathbf{q}^2+\delta} - \frac{1}{\mathbf{q}^2+\tilde{\delta}}\right)$$
$$+ \bar{a}^2_{\mathrm{cex}} \int d^3p\, d^3q \left(\Psi^\dagger(\mathbf{p}-\mathbf{q}) - \Psi^\dagger(\mathbf{p})\right)\Psi(\mathbf{p})\left(\frac{1}{\mathbf{q}^2+\rho} - \frac{1}{\mathbf{q}^2+\tilde{\delta}}\right)$$

$$\Delta a^{(2)} = \bar{a}^2_{\mathrm{cex}} \int d^3q \left(\frac{1}{\mathbf{q}^2+\tilde{\delta}} - \frac{1}{\mathbf{q}^2+\tilde{\delta}+2M_\pi\Delta_{\mathrm{N}}-\Delta_\pi}\right)$$

$$\bar{a}^2 = \frac{\xi_{\mathrm{p}}^2}{\pi^2\xi_{\mathrm{d}}}\left(2(a^-)^2 + 2a^-\Delta a^- - \sqrt{2}\,a^-\Delta a^{\mathrm{cex}}_{\pi^-p}\right), \quad \bar{a}^2_{\mathrm{cex}} = \frac{\xi_{\mathrm{p}}^2}{\pi^2\xi_{\mathrm{d}}}\left((a^-)^2 - \sqrt{2}\,a^-\Delta a^{\mathrm{cex}}_{\pi^-p}\right)$$

$\xi_{\mathrm{p}} = 1 + M_\pi/m_{\mathrm{p}}, \quad \xi_{\mathrm{d}} = 1 + M_\pi/m_{\mathrm{d}}$

Numerical results (in $10^{-3}M_\pi^{-1}$):

a^{static}	$a^{\mathrm{static}}_{\mathrm{NLO}}$	a^{cut}	$\Delta a^{(2)}$	a^{triple}	$a^{\pi\pi}$
-24.1 ± 0.7	3.8 ± 0.2	-4.8 ± 0.5	0.2	2.6 ± 0.5	-0.2 ± 0.3

Pion–nucleon σ term

$$\bar{u}(p')\sigma_{\pi N}(t)u(p) = \langle N(p')|\hat{m}(\bar{u}u + \bar{d}d)|N(p)\rangle, \qquad \sigma_{\pi N} \equiv \sigma_{\pi N}(0)$$

$$\left(\frac{m_{\rm s}}{\hat{m}} - 1\right)(1-y)\sigma_{\pi N} = \frac{m_{\rm s} - \hat{m}}{2m_{\rm p}} \underbrace{\langle N(p)|\bar{u}u + \bar{d}d - 2\bar{s}s|N(p)\rangle}_{\text{from } SU(3) \text{ mass difference}}$$

$$y = \frac{2\langle N(p)|\bar{s}s|N(p)\rangle}{\langle N(p)|\bar{u}u + \bar{d}d|N(p)\rangle}$$