$\pi\pi$ scattering from low to high energy

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MESON 2010, Krakow, June 15. 2010

Outline

Introduction

Low energy

High energy*
- Phenomenological inputs
- *D* and *F* waves
- Constraints on high-energy behaviour

Summary

* Work in progress together with Irinel Caprini and Heiri Leutwyler

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Why is $\pi\pi$ scattering interesting

- pions $=$ Goldstone bosons of spontaneous χSB of QCD
- $m_{u,d}/\Lambda_{\text{QCD}} \sim$ percent $\Rightarrow$ high precision possible
- S-matrix approach: $\pi\pi$ scattering amplitude related only to itself, also in crossed channels ($s < s_{\text{inel}}$)
- the two scattering lengths (subtractions constants) are the essential parameters at low energy
- conversely, many other observables are influenced by the $\pi\pi$ interaction in intermediate or final states (*e.g.* $K \to 2\pi, 3\pi$, $\eta \to 3\pi$, $(g-2)_\mu$, $\pi N \to \pi N$, $pp \to pp\pi^+\pi^-$, etc.)

see talks by Kupść, Lebiedowicz

Chiral symmetry + Roy equations

Approach adopted here:

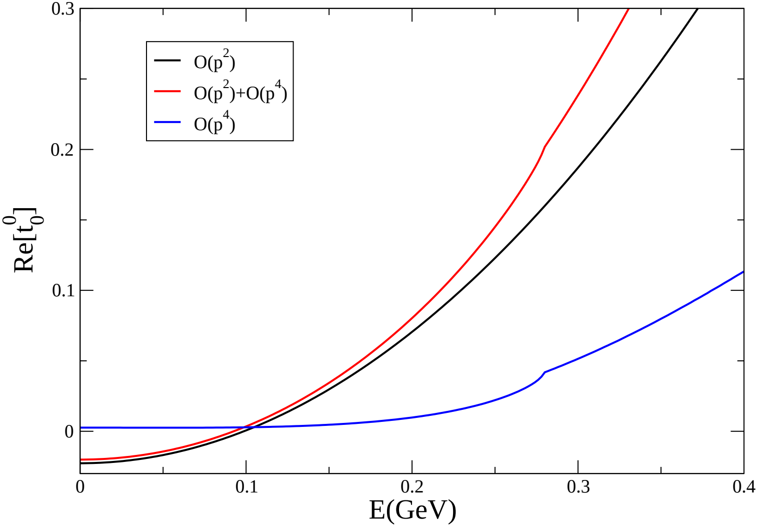
- ChPT:
  expansion of $A(\pi\pi \to \pi\pi)$ in powers of m_q and p

- dispersion relations:
  - exact mathematical condition
  - only two free parameters at low energy

- $\Rightarrow$ ChPT fixes the two subtraction constants

  $\Rightarrow$ Roy equation solutions: amplitude at any energy

Chiral symmetry + Roy equations



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Roy equations

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

S.M. Roy (71)

$$\begin{aligned}
\mathrm{Re}\, t_0^0(s) &= k_0^0(s) + \int_{4M_\pi^2}^{s_0} ds' K_{0\,0}^{0\,0}(s,s')\, \mathrm{Im}\, t_0^0(s') \\
&+ \int_{4M_\pi^2}^{s_0} ds' K_{0\,1}^{0\,1}(s,s')\, \mathrm{Im}\, t_1^1(s') \\
&+ \int_{4M_\pi^2}^{s_0} ds' K_{0\,0}^{0\,2}(s,s')\, \mathrm{Im}\, t_0^2(s') + f_0^0(s) + d_0^0(s) \\
k_0^0(s) &= a_0^0 + \frac{s-4M_\pi^2}{12M_\pi^2}(2a_0^0 - 5a_0^2) \\
f_0^0(s) &= \sum_{I'=0}^{2}\sum_{\ell'=0}^{1}\int_{s_0}^{s_3} ds' K_{0\ell'}^{0I'}(s,s')\, \mathrm{Im}\, t_{\ell'}^{I'}(s') \\
d_0^0(s) &= \text{all the rest}
\end{aligned}$$

$[\sqrt{s_0} = 0.8\mathrm{GeV} \quad \sqrt{s_3} = 2\mathrm{GeV}]$

Roy equations

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

S.M. Roy (71)

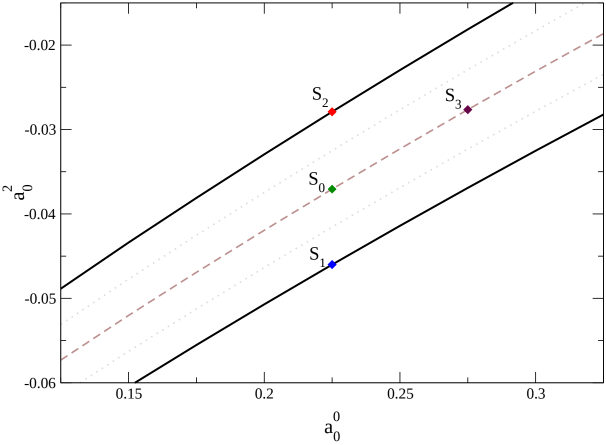
Numerical solutions of the Roy equations

Pennington-Protopopescu, Basdevant-Froggatt-Petersen (70s)
Ananthanarayan, GC, Gasser and Leutwyler (00)
Descotes-Genon, Fuchs, Girlanda and Stern (01)
Kamiński, Peláez and Ynduráin (08)

Input: S- and P-wave imaginary parts above 0.8 GeV
imaginary parts of all higher waves
two subtraction constants, *e.g.* a_0^0 and a_0^2

Output: the full $\pi\pi$ scattering amplitude below 0.8 GeV
Note: a_0^0, a_0^2 inside the universal band $\Rightarrow$ the solution is unique

Numerical solutions



Ananthanarayan, GC, Gasser, Leutwyler (00)

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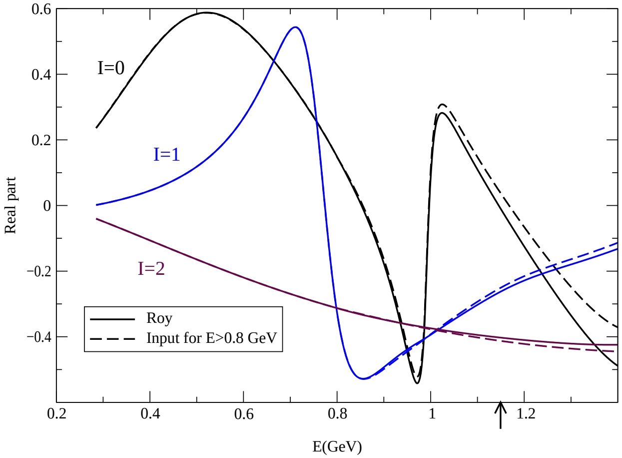

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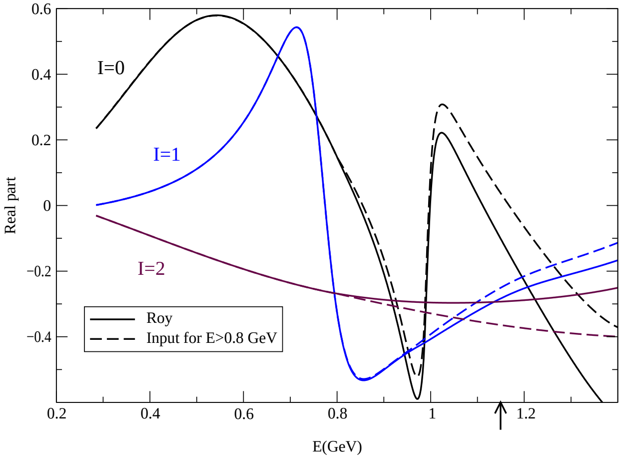

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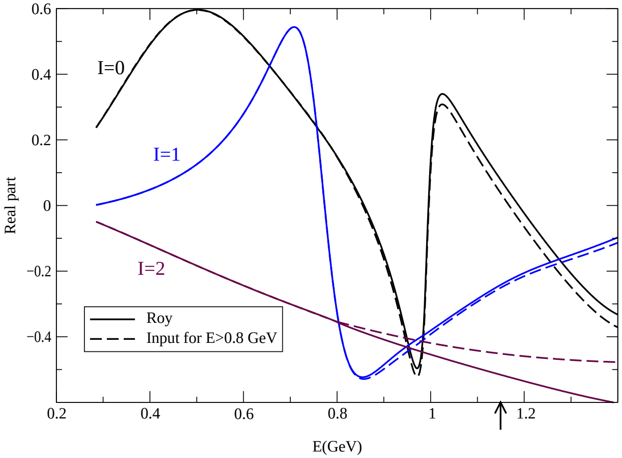

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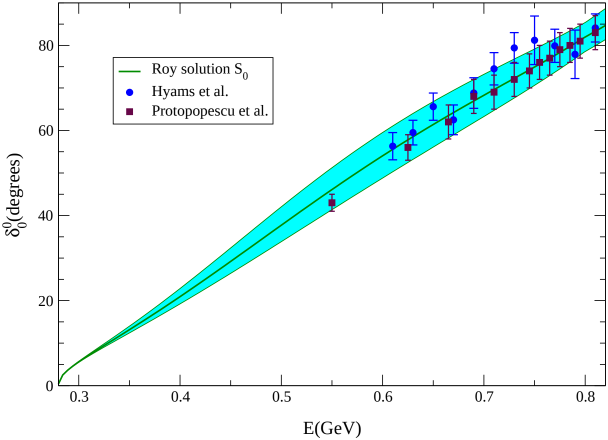
Roy+ChPT: final results

GC, Gasser and Leutwyler (01)

Phase shifts:



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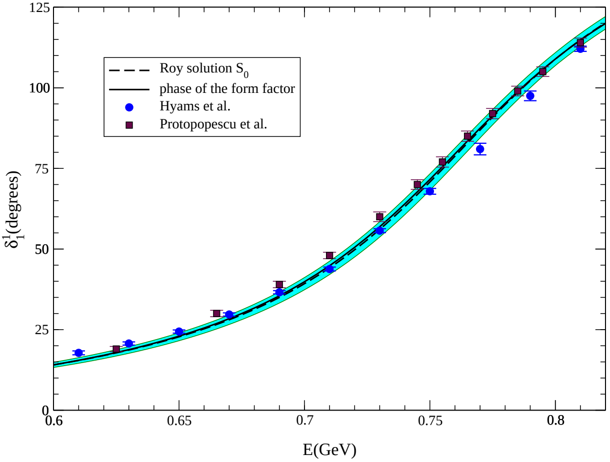

Roy+ChPT: final results

GC, Gasser and Leutwyler (01)

Scattering lengths

$$
\begin{aligned}
a_0^0 &= \quad 0.220 \pm 0.001 + 0.009\Delta\ell_4 - 0.002\Delta\ell_3 \\
10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.01\Delta\ell_4 - 0.004\Delta\ell_3
\end{aligned}
$$

where $\bar{\ell}_4 = 4.4 + \Delta\ell_4$ $\quad\bar{\ell}_3 = 2.9 + \Delta\ell_3$

Adding errors in quadrature $[\Delta\ell_4 = 0.2,\ \Delta\ell_3 = 2.4]$

$$
\begin{aligned}
a_0^0 &= \quad 0.220 \pm 0.005 \\
10 \cdot a_0^2 &= -0.444 \pm 0.01 \\
a_0^0 - a_0^2 &= \quad 0.265 \pm 0.004
\end{aligned}
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Roy+ChPT: final results

GC, Gasser and Leutwyler (01)

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Adding errors in quadrature $[\Delta\ell_4 = 0.2,\ \Delta\ell_3 = 2.4]$

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Peláez and Ynduráin have criticized these results

Claim 1: our input above 1.4 GeV is not correct (PY 03)

The criticism has been answered (Caprini *et al.* 03)

Roy+ChPT: final results

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Scattering lengths

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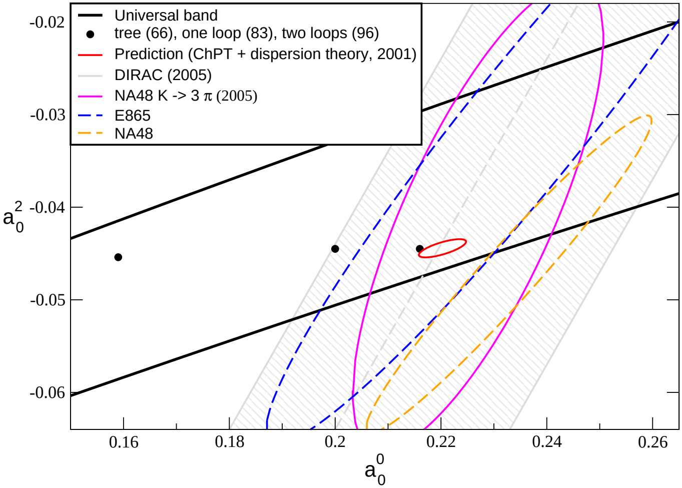
Peláez and Ynduráin have criticized these results

Claim 2: our calculation for $\langle r^2 \rangle_s$ is not correct (Y, 04)

The criticism has been answered (Ananthanarayan *et al.* 04)

Experimental tests



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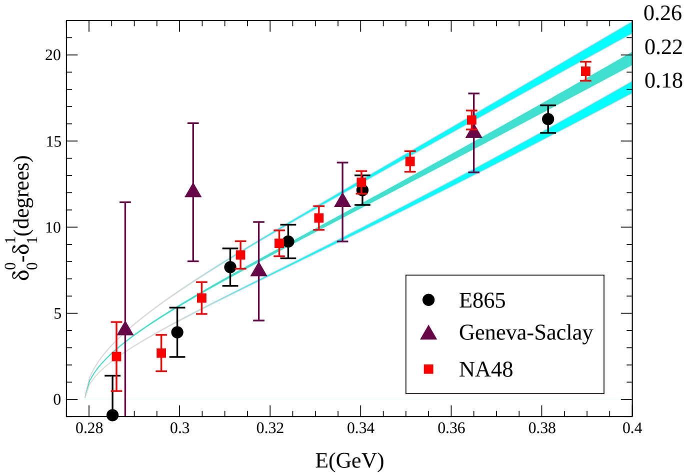

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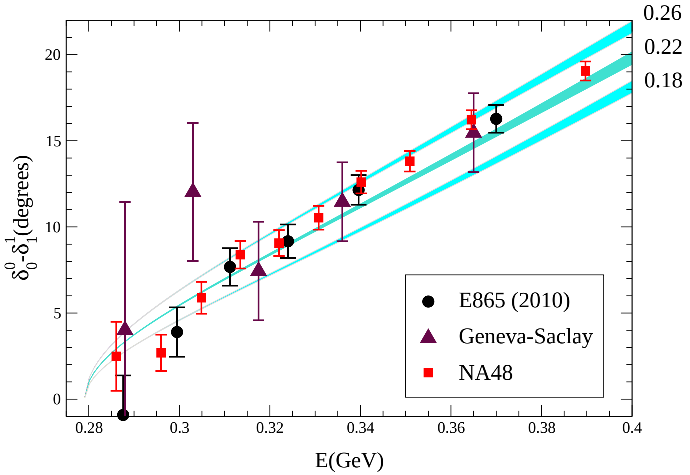
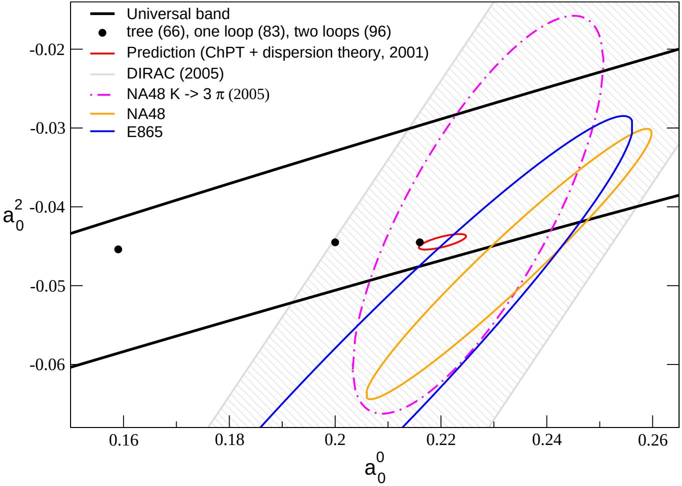


Recent update: E865 corrected their data

Experimental tests



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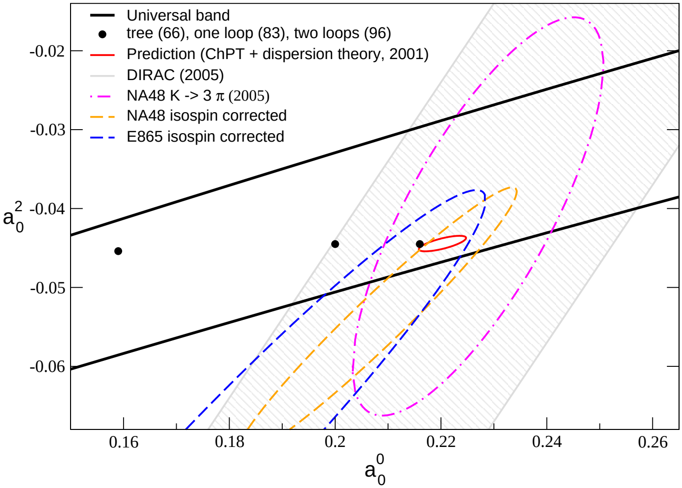

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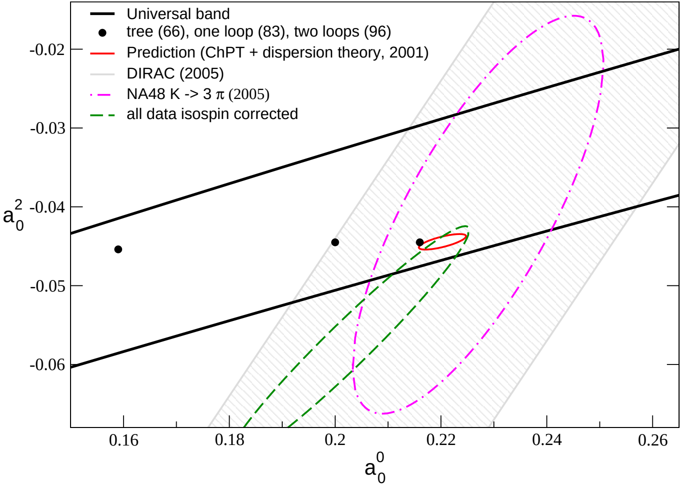

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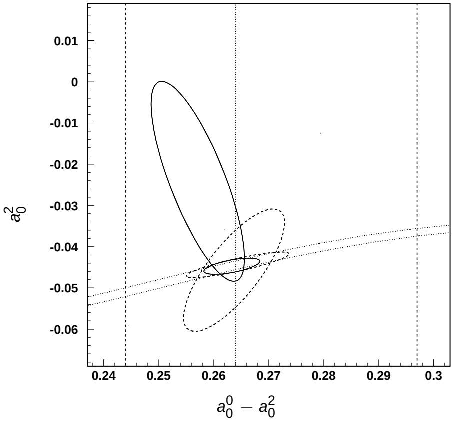

Figure from NA48/2 Eur.Phys.J.C64:589,2009

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Extensions and improvements

The ACGL Roy analysis can be extended/improved:

- High energy part (Regge) had been taken from the literature
  - new information has become available (*e.g.* Compete)
  - various sum rules put constraints on Regge (considered only partially in ACGL)
- *D* and *F* waves ($\Rightarrow$ driving terms) taken from the literature
  Roy equations can be solved for them too
- Roy equations valid up to $68M_\pi^2 \sim (1.15\text{GeV})^2$
  region $0.8 < \sqrt{s} < 1.15$ GeV can be constrained further
- more data available after 2001 ($\pi N \to \pi\pi N$ with polarized targets Kamiński, Lesniak and Rybicki) and ($e^+e^- \to \pi^+\pi^-$ cross section, CMD-2, SND, KLOE, and more recently BABAR)

Roy equations extended: impact of $s_0 \to s_1$

$$
\begin{aligned}
\mathrm{Re}\, t_0^0(s) &= k_0^0(s) + \int_{4M_\pi^2}^{s_1} ds' K_{0\,0}^{0\,0}(s,s')\, \mathrm{Im}\, t_0^0(s') \\
&+ \int_{4M_\pi^2}^{s_1} ds' K_{0\,1}^{0\,1}(s,s')\, \mathrm{Im}\, t_1^1(s') \\
&+ \int_{4M_\pi^2}^{s_1} ds' K_{0\,0}^{0\,2}(s,s')\, \mathrm{Im}\, t_0^2(s') + f_0^0(s) + d_0^0(s) \\
k_0^0(s) &= a_0^0 + \frac{s-4M_\pi^2}{12M_\pi^2}(2a_0^0 - 5a_0^2) \\
f_0^0(s) &= \sum_{I'=0}^{2} \sum_{\ell'=0}^{1} \int_{s_1}^{s_2} ds' K_{0\ell'}^{0I'}(s,s')\, \mathrm{Im}\, t_{\ell'}^{I'}(s') \\
d_0^0(s) &= \text{all the rest}
\end{aligned}
$$

$$\sqrt{s_1} = 1.15\mathrm{GeV} \quad \sqrt{s_2} \sim 1.7\mathrm{GeV}$$

Roy equations extended: impact of $s_0 \to s_1$

Multiplicity of the solution $\Leftrightarrow$ value of the phases at the matching point: Epele, Wanders (77), Gasser, Wanders (99)

- $\sqrt{s_0} = 0.8$ GeV $\Rightarrow$ unique solution (ACGL)
- $\sqrt{s_1} = 1.15$ GeV $\Rightarrow$ 3 free parameters

If $s_1 > s_{\rm inel} \Rightarrow$ need input on $\eta^I_\ell(s)$

Free parameters

Input phases – need to know: [$\sqrt{s_0} = 0.8$ GeV, $\sqrt{s_1} = 1.15$ GeV]

- three input phases for the $S0$ wave:

$$\delta_0^0(s_0) = \begin{cases} 82.3^\circ \pm 3.4^\circ & \text{narrow range (ACGL 00)} \\ 82.3^\circ \, {}^{+10^\circ}_{-4^\circ} & \text{broad range (CCL 06)} \end{cases}$$

$$\delta_0^0(4M_K^2) = 185^\circ \pm 10^\circ$$

$$\delta_0^0(s_1) = 260^\circ \pm 10^\circ$$

- two input phases for the P wave

$$\delta_1^1(s_0) = (108.9 \pm 2)^\circ$$

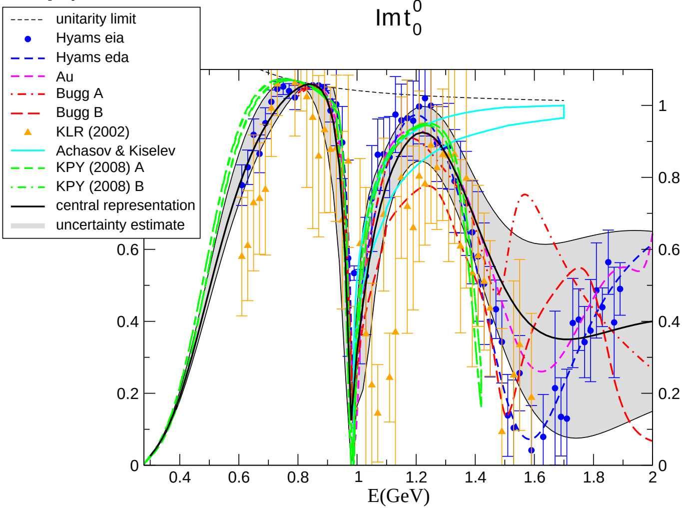
$$\delta_1^1(s_1) = (166.5 \pm 2)^\circ$$

Conservative range: $e^+e^- \to \pi^+\pi^-$ data more precise

- no input phase for the $S2$ wave: $a_0^2 \Rightarrow \delta_0^2(s_1)$

Inputs taken from phenomenology

Imaginary parts:



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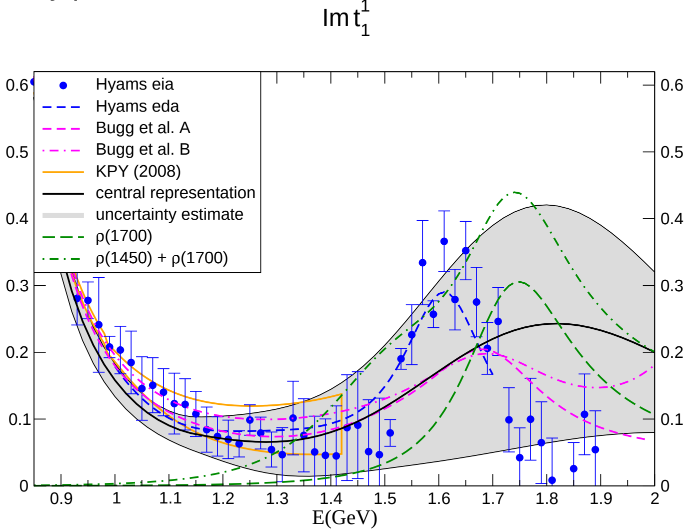
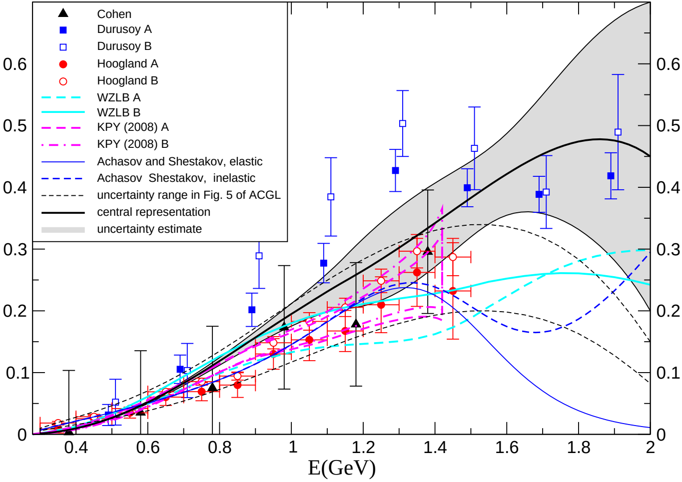


Inputs taken from phenomenology

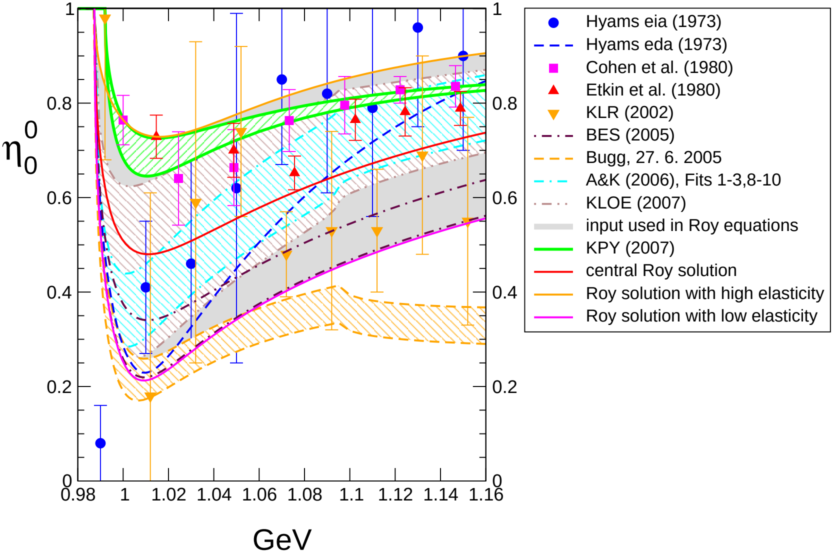
Imaginary parts:

$\mathrm{Im}\,t_0^2$

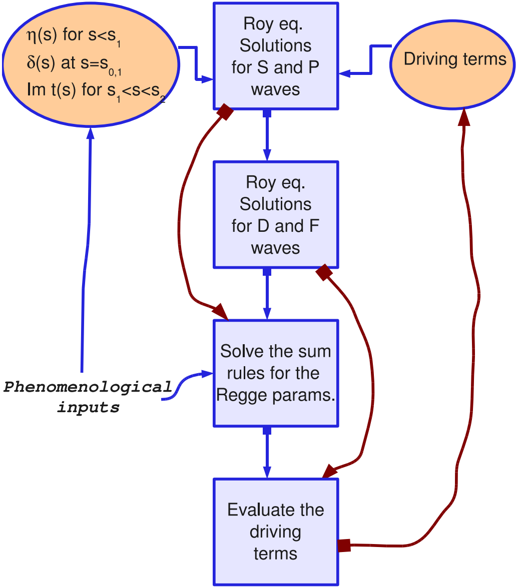


Inputs taken from phenomenology

Inelasticities:



Flowchart of the analysis



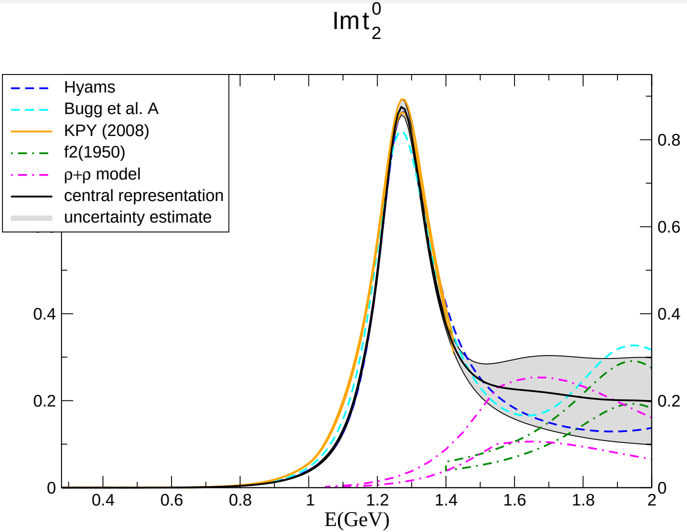
Roy equations for D and F waves

$$
\begin{aligned}
\mathrm{Re}\, t_2^0(s) &= + \fint_{4M_\pi^2}^{s_1} ds' K_{2\,2}^{0\,0}(s,s')\,\mathrm{Im}\, t_2^0(s') \\
&+ \int_{4M_\pi^2}^{s_1} ds' K_{2\,3}^{0\,1}(s,s')\,\mathrm{Im}\, t_3^1(s') \\
&+ \int_{4M_\pi^2}^{s_1} ds' K_{2\,2}^{0\,2}(s,s')\,\mathrm{Im}\, t_2^2(s') + f_2^0(s) + d_2^0(s) \\
f_2^0(s) &= \sum_{I'=0}^{2} \sum_{\ell'=0}^{1} \fint_{s_1}^{s_2} ds' K_{2\ell'}^{0I'}(s,s')\,\mathrm{Im}\, t_{\ell'}^{I'}(s') \\
d_2^0(s) &= \text{S, P, G and higher waves, high energy}
\end{aligned}
$$

Here the “driving terms” dominate the rhs at low energy: the S and P wave contributions fix to a large extent the D, F and higher waves

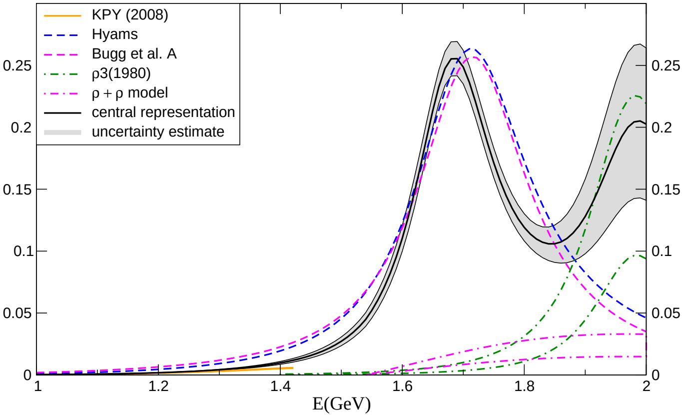
see also poster by R. Kamiński

Roy equations for *D* and *F* waves



Roy equations for *D* and *F* waves

$\mathrm{Im}\,t_3^1$



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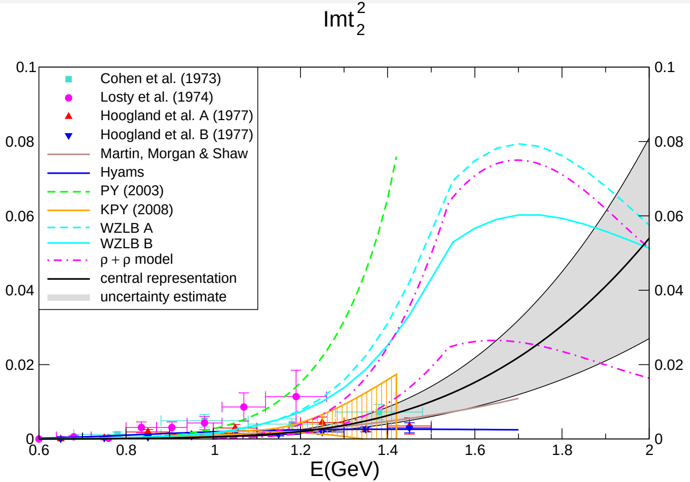

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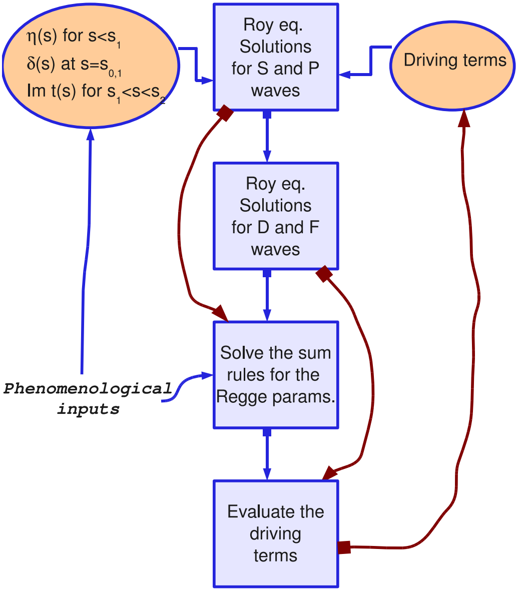

Regge representation

Regge formulae for imaginary parts at fixed I_t

$$\begin{aligned}
\mathrm{Im}\, T^{I_t=0}(s,t) &= \beta_P(t)\left(\frac{s}{\bar{s}}\right)^{\alpha_P(t)} + B\log^2(s/s_B) + \beta_f(t)\left(\frac{s}{\bar{s}}\right)^{\alpha_f(t)} \\
\mathrm{Im}\, T^{I_t=1}(s,t) &= \beta_\rho(t)\left(\frac{s}{\bar{s}}\right)^{\alpha_\rho(t)} \\
\mathrm{Im}\, T^{I_t=2}(s,t) &= \beta_e(t)\left(\frac{s}{\bar{s}}\right)^{\alpha_e(t)}
\end{aligned}$$

COMPETE collaboration: phenomenological determination of these parameters

Peláez and Ynduráin have also determined these parameters independently, specifically for $\pi\pi$ scattering

Sum rules and asymptotic behaviour

Roy equations do not account for all known constraints:

- in the $I_t = 1$ channel, one subtraction less is necessary
  $\Rightarrow$ Olsson sum rule

$$2a_0^0 - 5a_0^2 = \frac{M_\pi^2}{8\pi^2} \int_{4M_\pi^2}^{\infty} ds \frac{2\,\mathrm{Im}\,T^0(s,0) + 3\,\mathrm{Im}\,T^1(s,0) - 5\mathrm{Im}\,T^2(s,0)}{s\,(s-4M_\pi^2)}$$

- extend the sum rule to any $t \leq 0$

$$\int_{4M_\pi^2}^{\infty} ds\, \frac{2\,\mathrm{Im}\,\bar{T}^0(s,t) + 3\,\mathrm{Im}\,\bar{T}^1(s,t) - 5\,\mathrm{Im}\,\bar{T}^2(s,t)}{12\,s\,(s+t-4M_\pi^2)} - \int_{4M_\pi^2}^{\infty} ds\, \frac{(s-2M_\pi^2)\,\mathrm{Im}\,T^1(s,0)}{s\,(s-4M_\pi^2)\,(s-t)\,(s+t-4M_\pi^2)} = 0$$

- crossing symmetry not fully implemented

$\Rightarrow$ one t-dependent sum rule in each I_t channel

S and *P* waves do not enter these

Regge parameters

Our approach:

- the trajectories $\alpha_i(t)$ and some residues (*e.g.* $\beta_P(0)$) are well known phenomenologically: [α' values in GeV^{-2}]

$\beta_P(0) = 94 \pm 1$	$\beta_f = 69 \pm 2$	$\bar{B} = 0.025 \pm 0.001$	
$\alpha_P(0) = 1$	$\alpha_f(0) = 0.54 \pm 0.05$	$\alpha_\rho(0) = 0.45 \pm 0.02$	$\alpha_e(0) = 0$
$\alpha'_P(0) = 0.25 \pm 0.05$	$\alpha'_f(0) = 0.90 \pm 0.05$	$\alpha_\rho(0) = 0.91 \pm 0.02$	$\alpha'_e(0) = 0.5 \pm 0.1$

- the low-energy contribution to the integrals are determined by the solution to the Roy equations

$\Rightarrow$ use the sum rules to determine the unknown residues $\beta_i(t)$

Example: Olsson sum rule

$$2a_0^0 - 5a_0^2 = \frac{M_\pi^2}{8\pi^2}\int_{4M_\pi^2}^{s_2} ds\ [\text{partial w.}] + \beta_\rho(0)\frac{3M_\pi^2}{4\pi^2}\int_{s_2}^{\infty} ds\,\frac{(s/\bar{s})^{\alpha_\rho(0)}}{s(s-4M_\pi^2)}$$

Regge parameters

- In this way we tune the Regge residues such that the integrals below and above ~ 1.7 GeV match exactly
- Moreover we also make sure that the imaginary parts (cross sections) are continuous at ~ 1.7 GeV
- In order to do this we multiply the Regge representations with "preasymptotic terms":

$$\mathrm{Im}\,T^{I_t}(s,t) = \mathrm{Im}\,T^{I_t}_{\mathrm{Regge}}(s,t)\left(1 + r_{I_t}\frac{\bar{s}}{s}\right)$$

and tune the parameter r_{I_t} accordingly
e.g. for $I_t = 1$ we get:

$$\beta_\rho(0) = 97 \pm 13 \qquad r_1 = -1.4 \pm 0.5$$

Regge parameters

- In this way we tune the Regge residues such that the integrals below and above $\sim$ 1.7 GeV match exactly
- Moreover we also make sure that the imaginary parts (cross sections) are continuous at $\sim$ 1.7 GeV
- In order to do this we multiply the Regge representations with “preasymptotic terms”:

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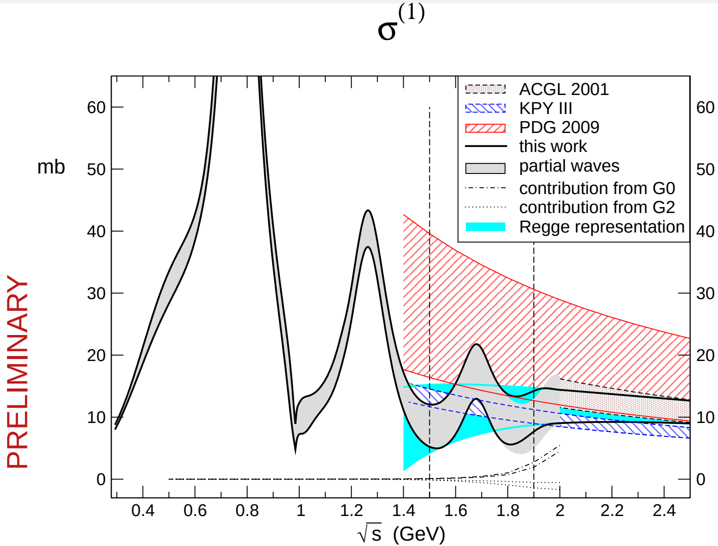
and tune the parameter r_{I_t} accordingly

- We also fix the t-dependence of the residues (profile) by continuity

$$\beta_X(t) \equiv \beta_X(0)b_X(t)$$

Regge parameters



Work in progress with I. Caprini and H. Leutwyler

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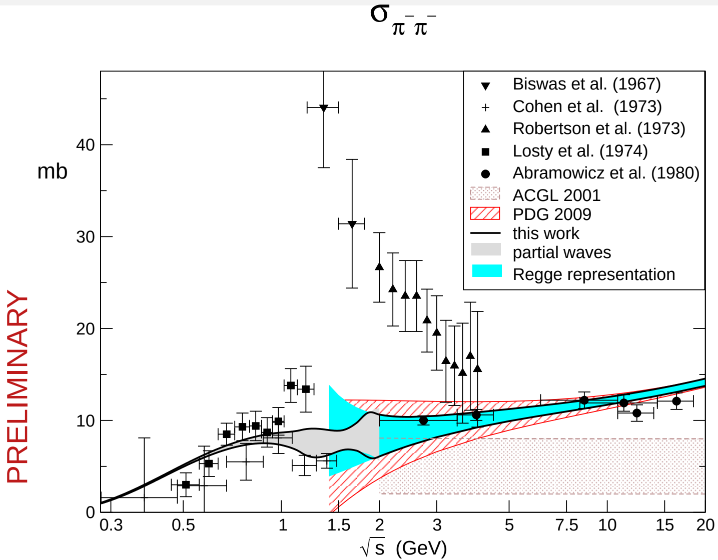

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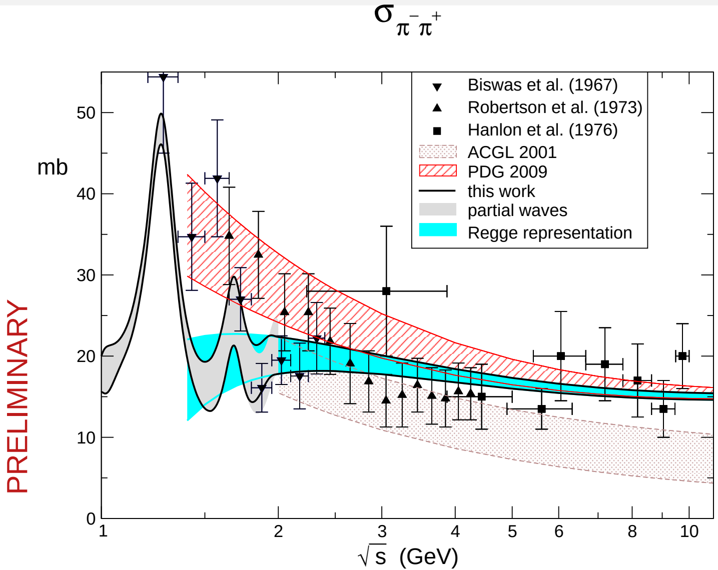

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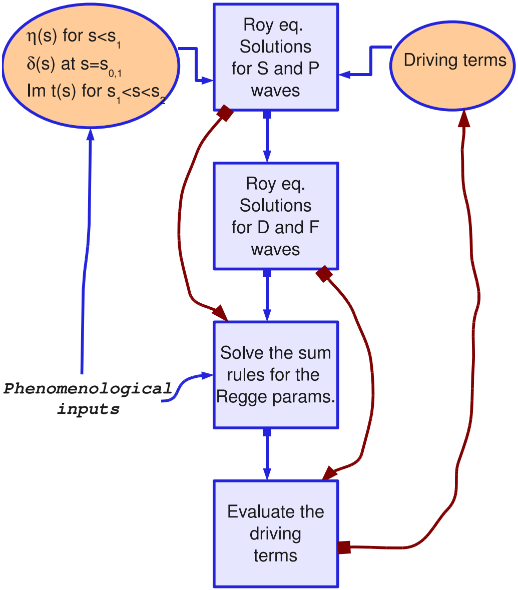
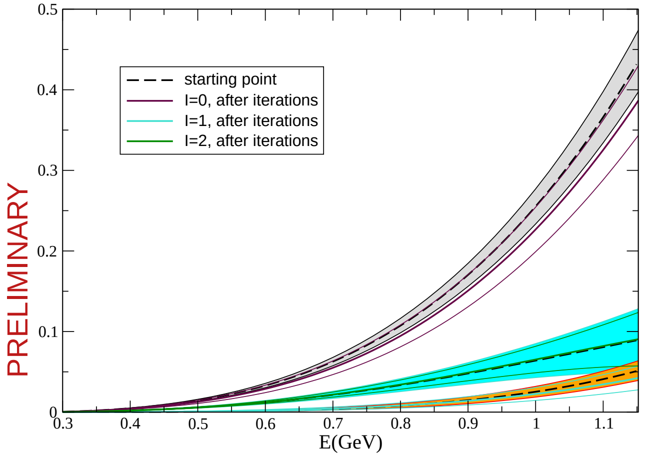


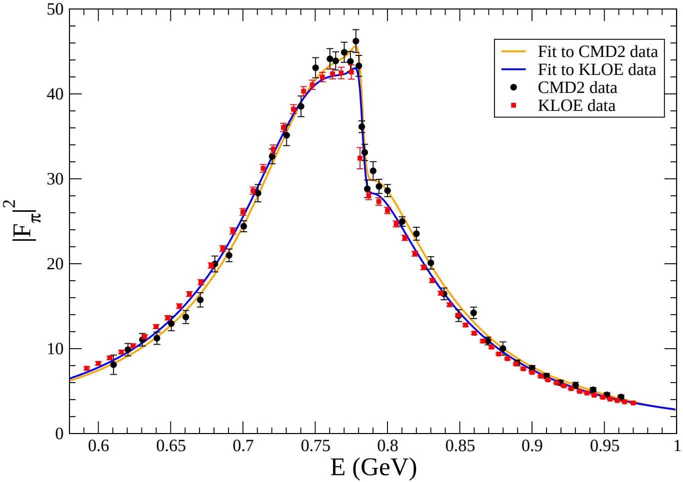
Driving terms

The iterative determination of the driving terms converges immediately:



Work in progress with I. Caprini and H. Leutwyler

Application: fit to the vector form factor



$$F_\pi(s) \propto \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right]$$

Outline

Introduction

Low energy

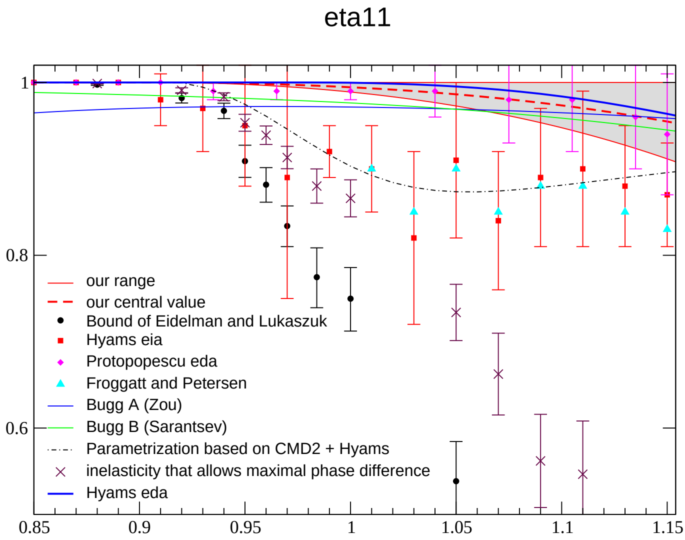
High energy*
- Phenomenological inputs
- *D* and *F* waves
- Constraints on high-energy behaviour

Summary

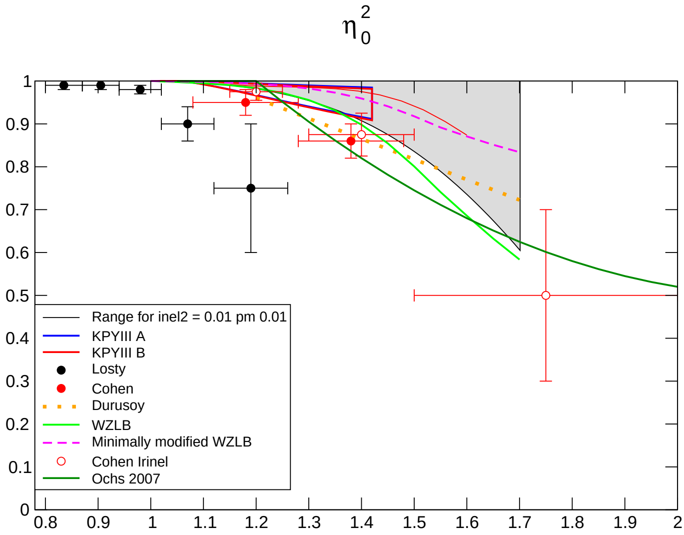
Summary

- the $\pi\pi$ scattering amplitude at low energy can be predicted with high accuracy thanks to a combination of chiral symmetry and dispersion relations
- experiments (E865, DIRAC and NA48) are reaching the same level of accuracy and confirm the theory predictions
- I have presented an extension of the Roy equation analysis to higher energy and higher partial waves
- no significant changes at low energy ($<$ 0.8 GeV), but a much better control on the high-energy part
- this provides essential information to analyses of other processes where $\pi\pi$ scattering plays a role (*e.g.* $\eta \to 3\pi$ or $(g-2)_\mu$, $pp \to pp\pi\pi$)

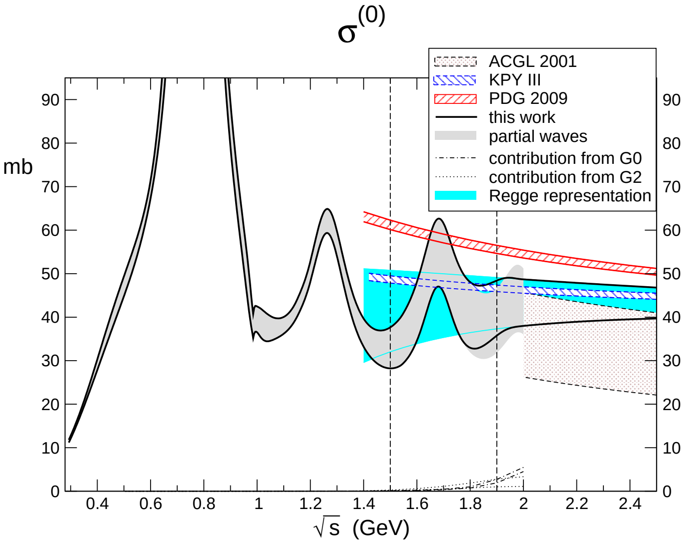
Inelasticities



Inelasticities



Cross sections



Cross sections